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# Physical Interpretation of the Reactive Power in Terms of CPC Power Theory Revisited

**Dimitri JELTSEMA, Jacob VAN DER WOUDE**

*Department of Mathematical Physics of the Delft Institute of Applied Mathematics, Delft University of Technology, The Netherlands*
*d.jeltsema@tudelft.nl; j.w.vanderwoude@tudelft.nl*

**Summary: Recently the authors have developed a time-domain representation of the Currents' Physical Components (CPC) power theory. Consequently, piece-wise continuous currents and voltages, such as square waves and sawtooth signals, can naturally be taken into account without approximating them by a finite number of harmonics. Based on the time-domain CPC approach, this paper refines and corrects some of the assertions made by Czarnecki in his paper 'Physical Interpretation of the Reactive Power in Terms of CPC Power Theory'.**

**Key words:**
*power theory, reactive power, current's physical components*

## 1. INTRODUCTION

In [1], the author argues that reactive power is not necessarily caused by energy oscillation. This assertion is motivated using a purely resistive load that is supplied by a sinusoidal source voltage and controlled by an ideal TRIAC, switched at firing angle α, as depicted in Fig. 1.

Under the assumption that the supply voltage equals

$$u(t) = 220\sqrt{2}\,\sin(\omega_f t)\ \mathrm{V}$$

a load resistance R=1Ω, and a firing angle α=135°, the current drawn from the source has the waveform depicted in Fig. 2. Since the supplied current has a root-mean-square (rms) value that is equal to $\|i(t)\|=66.307$ A, the apparent power equals $S=\|u(t)\|\,\|i(t)\|=14.588$ kVA. Furthermore, the active power is computed from

$$P_a = \frac{1}{T}\int_0^T u(t)\,i(t)\,dt = 4.397\,\mathrm{kW}$$

where $T = 2\pi/\omega_f$ is the period of the voltage supply. Now, since the power factor (PF) is defined as the ratio between active and apparent power, we have PF=0.301. This suggests that there must be a non-active power present in the circuit. Indeed, the author of [1] proceeds to decompose the supply current into a fundamental harmonic

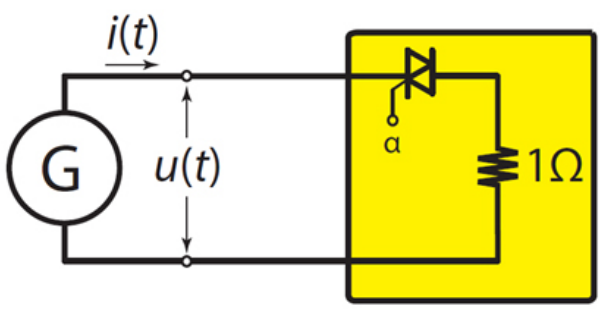


Fig. 1. Circuit with resistive load and TRIAC

$$i_1(t) = 40.317\sqrt{2}\,\sin(\omega_f t - 60.28°)\,\mathrm{A}$$

which is subsequently decomposed into an active component, $i_{1_a}(t)$, that is directly proportional (collinear) with the supply voltage and a reactive (quadrature) component $i_{1_r}(t)$ as

$$i_1(t) = 19.987\sqrt{2}\sin(\omega_f t) - 35.014\sqrt{2}\cos(\omega_f t)$$

In [1], the reactive current is computed as $i_{1_r}(t) = +35.014\sqrt{2}\cos(\omega_f t)$, which seems to be a typographic error.

In the analysis that follows, the author of [1] states that since there is no scattered current component, the supply current has only three physical orthogonal components, satisfying

$$\|i(t)\| = \|i_{1_a}(t)\|^2 + \|i_{1_r}(t)\|^2 + \|i_h(t)\|^2$$

where $i_h(t) = i(t) - i_1(t)$ represents the load generated harmonic current that is due to the TRIAC invoked distortion of the supply voltage. Consequently, the associated powers are computed as follows:

| $P_a$ [kW] | $D_s$ [kVA] | $Q_r$ [kVAr] | $D_h$ [kVA] | $S$ [kVA] |
|---|---|---|---|---|
| 4.397 | 0 | 7.703 | 11.581 | 14.588 |

Here, the reactive and harmonic powers are obtained from

$$Q_r = \|u(t)\|\,\|i_{1_r}(t)\| \qquad D_h = \|u(t)\|\,\|i_h(t)\|$$

respectively.

## 2. CONTRIBUTION

The main contribution of the present paper is to show that the decomposition outlined above does not coincide with the general CPC framework presented in [4]. Moreover, it will be shown that there actually is a scattered component in the supply current and, consequently, the scattered power $D_s \neq 0$. Furthermore, the reactive power is shown to possess more than one harmonic. These properties come easily into the picture when the circuit of Fig. 1 is analysed from a purely time-domain perspective of CPC recently proposed by the

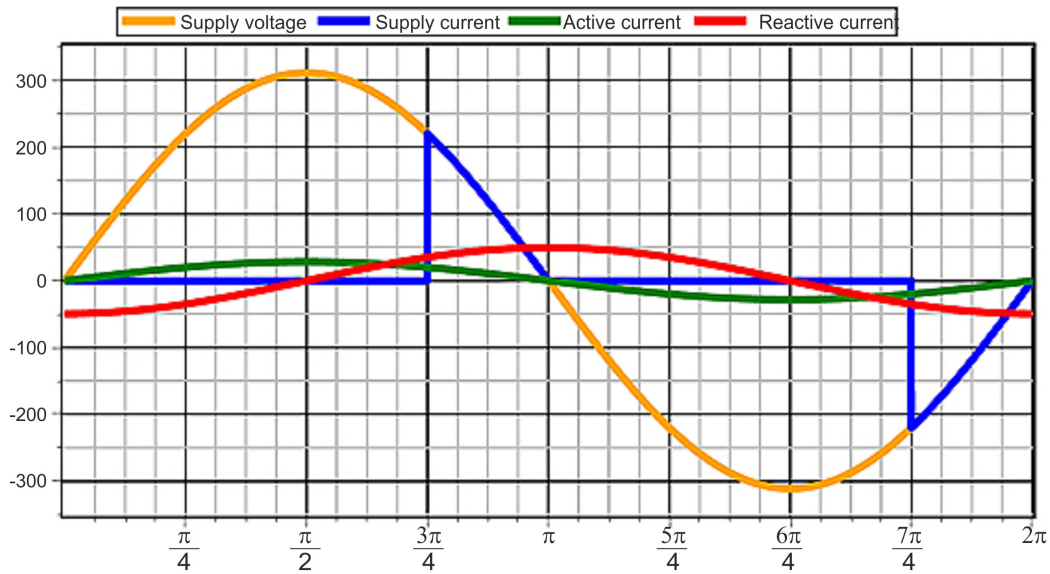


Fig. 2. Voltage and current waveforms for the TRIAC circuit

authors in [3]. For ease of reference, we briefly recall the basics of the conventional CPC method first.

## 3. THE CPC METHOD

The development of the CPC-based power theory dates back to 1984 [2], with explanations of power properties in single-phase circuits driven by a non-sinusoidal voltage source of the form

$$u(t) = U_0 + \sqrt{2}\,\mathrm{Re}\left\{\sum_{n\in N} Y_n U_n e^{jn\omega_f t}\right\}$$

where $N$ is the set of harmonics present in the signal and $\omega_f$ is the fundamental frequency. The basic assumption is that the load is linear time-invariant (LTI), which can be characterized by a frequency-dependent admittance of the form

$$Y(\omega) = G(\omega) + j\,B(\omega) \tag{1}$$

Thus, for each harmonic $n \in N$, the associated admittance can be written as $Y_n = G_n + j\,B_n$. Consequently, the load current can be expressed as

$$i(t) = Y_0 U_0 + \sqrt{2}\,\mathrm{Re}\left\{\sum_{n\in N} Y_n U_n e^{jn\omega_f t}\right\}$$

The main idea then is to decompose the latter into three orthogonal components as

$$i(t) = i_a(t) + i_s(t) + i_r(t) \tag{2}$$

The first component is the *active* current, and can be written as

$$i_a(t) = G_e u(t) = G_e U_0 + \sqrt{2}\,\mathrm{Re}\left\{\sum_{n\in N} G_e U_n e^{jn\omega_f t}\right\}$$

where $G_e$ denotes the equivalent conductance

$$G_e = \frac{\langle u(t), i(t)\rangle}{\|u(t)\|^2} \tag{3}$$

The remaining components can be found by extracting the active current from the load current, i.e.,

$$i_F(t) = i(t) - i_a(t) =$$

$$= (Y_0 - G_e)U_0 + \sqrt{2}\,\mathrm{Re}\left\{\sum_{n\in N}(Y_n - G_e)U_n e^{jn\omega_f t}\right\} =$$

$$= (Y_0 - G_e)U_0 + \sqrt{2}\,\mathrm{Re}\left\{\sum_{n\in N}(G_n + j\,B_n - G_e)U_n e^{jn\omega_f t}\right\}$$

and decomposing the latter into

$$i_s(t) := (G_0 - G_e)U_0 + \sqrt{2}\,\mathrm{Re}\left\{\sum_{n\in N}(Y_n - G_e)U_n e^{jn\omega_f t}\right\}$$

$$i_r(t) := \sqrt{2}\,\mathrm{Re}\left\{\sum_{n\in N} j\,B_n U_n e^{jn\omega_f t}\right\},$$

referred to as the *scattered* and *reactive* current, respectively.

Although these current components are in itself not physical quantities, they can be associated to three distinctive physical phenomena in the load:

1) Permanent energy conversion—active current $i_a(t)$;
2) Change of load conductance $G_n$ with harmonic order—scattered current $i_s(t)$;
3) Phase-shift between the voltage and current harmonics—reactive current $i_r(t)$.

For more details and a proof of orthogonality, the interested reader is referred to [4]. Note that the CPC

method uses techniques from both the time-domain and the frequency domain, and can therefore be considered as a *hybrid* approach.

## 4. CPC IN THE TIME-DOMAIN

Consider a single-input single-output (SISO) linear time-invariant (LTI) system of the form

$$dx(t)/dt = \mathcal{A}\,x(t) + \mathcal{B}\,u(t)$$

$$y(t) = \mathcal{C}\,x(t) + \mathcal{D}\,u(t)$$

where $x(t)$ denotes the state of the system, and the input $u(t)$ and the output $y(t)$ form a so-called power conjugated input-output pair, i.e., their product equals (instantaneous) power. If the input represents a voltage, then the output necessarily represent a current, and vice-versa. The matrices $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ are constant matrices of appropriate dimensions reflecting the network structure of the circuit. Under the assumption that the system is stable, and some integrable (periodic) input $u(t)$ for $t \geq 0$, the (stationary or steady-state) solution of the system (4) is given by

$$y^*(t) = \int_{-\infty}^{\infty} h(s)u(t-s)\,ds \tag{5}$$

where the function $h(t) := \mathcal{C}e^{\mathcal{A}t}\mathcal{B} + \mathcal{D}\delta(t)$ is assumed to be causal, i.e., $h(t) = 0$ for $t < 0$. Here $\delta(t)$ represents the Dirac delta function and $h(t)$ is called the impulse response. As any (real or complex) function can be written as a unique sum of even and odd functions, we decompose the impulse response into

$$h_e(t) := \frac{h(t) + h(-t)}{2}$$

$$h_0(t) := \frac{h(t) - h(-t)}{2} \tag{6}$$

satisfying $h(t) = h_e(t) + h_o(t)$.

### A. Hybrid Versus Time-Domain CPC

The rationale behind the even-odd decomposition (6) is that, under the condition that $h(t)$ is real and causal (a property that is generally satisfied in physical systems), the even and odd parts, $h_e(t)$ and $h_o(t)$, uniquely correspond to their real and imaginary counterparts parts in the complex frequency-domain, respectively. This one-to-one correspondence between the time-domain and the frequency-domain is known as the Kramers-Kronig relationship [6], [7]. Hence, if $u(t)$ represents the source voltage and $y(t)$ represents the corresponding source current, the even and odd part of the impulse response uniquely corresponds to the real and imaginary parts of the complex load admittance (1), respectively, i.e.,

$$h_e(t) \leftrightarrow G(\omega), \qquad h_o(t) \leftrightarrow B(\omega)$$

A pictorial representation of the Kramers-Kronig relations is shown in Fig. 3, whereas an elementary proof can be found in [8].

### B. Active, Scattered, and Reactive Power

Based on (6), the stationary response of the output can be decomposed into

$$\begin{aligned} y_{\parallel}^*(t) &:= \int_{-\infty}^{\infty} h_e(s)u(t-s)\,ds = \\ &= \int_0^{\infty} \frac{h(s)u(t-s)}{2}\,ds + \int_{-\infty}^{0} \frac{h(-s)u(t-s)}{2}\,ds \end{aligned} \tag{7}$$

and

$$\begin{aligned} y_{\perp}^*(t) &:= \int_{-\infty}^{\infty} h_o(s)u(t-s)\,ds = \\ &= \int_0^{\infty} \frac{h(s)u(t-s)}{2}\,ds - \int_{-\infty}^{0} \frac{h(-s)u(t-s)}{2}\,ds \end{aligned} \tag{8}$$

In relation to the input, $y_{\parallel}^*(t)$ corresponds to the in-phase part, whereas the out-of-phase part is represented by $y_{\perp}^*(t)$.

Based on this decomposition, we can now compute the active, reactive, and scattered components of the stationary

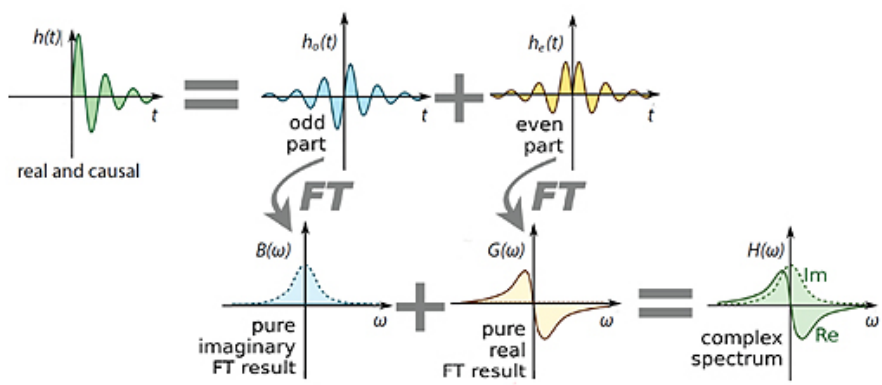


Fig. 3. The Kramers-Kronig relationship

output. The active and scattered components are contained in $y_{\|}^{*}(t)$ and can be extracted as

$$y_a^*(t) = \frac{\langle y_{\|}^*(t), u(t)\rangle}{\| u(t) \|^2} u(t) \tag{9}$$

and

$$y_s^*(t) = y_{\|}^*(t) - \frac{\langle y_{\|}^*(t), u(t)\rangle}{\| u(t) \|^2} u(t) = y_{\|}^*(t) - y_a^*(t) \tag{10}$$

respectively, whereas the reactive component is given by

$$y_r^*(t) = y_{\perp}^*(t) \tag{11}$$

The corresponding powers are then found as

$$P_a := \| u(t) \| \, \| y_a^*(t) \| \qquad Q_r := \| u(t) \| \, \| y_r^*(t) \|$$

$$D_s := \| u(t) \| \, \| y_s^*(t) \|$$

whereas for the apparent power we have

$$S := \| u(t) \| \, \| y^*(t) \|$$

## 5. THE TRIAC CIRCUIT REVISITED

Although the approach outlined in Section 4 is (so far) established for linear time-invariant (LTI) systems, the extension to static time-dependent circuits, like the TRIAC circuit of Fig. 1, can naturally be taking into consideration as follows. Since there are no inductors and capacitors in the circuit, the matrices $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are all void. Hence, the system (4) reduces to a static time-dependent input-output system of the form

$$y(t) = \mathcal{D}(t)\, u(t) \tag{12}$$

For the TRIAC circuit of Fig. 1, the input $u(t)$ and the output $y(t) = i(t)$ represent the supply voltage and current, respectively, whereas the direct feed-through term $\mathcal{D}(t)$ represents the load in the form of a time-varying linear conductor

$$\mathcal{D}(t) = \begin{cases} 0, & \text{if } t \in \left[\frac{kT}{2}, \frac{kT}{2} + \alpha\right), \; k = 0,1,2\ldots \\ 1, & \text{otherwise} \end{cases} \tag{13}$$

As the circuit does not exhibit any dynamics, it is sufficient to consider only the first period: $k = 0, 1$.

In order to apply a similar decomposition as in (6), the load characteristic over one period is split into a positive sequence

$$\mathcal{D}^+(t) = \begin{cases} 0, & \text{if } t \in [0, \alpha) \text{ and } t \in \left[\frac{T}{2}, \frac{T}{2} + \alpha\right) \\ 1, & \text{otherwise} \end{cases}$$

and a corresponding negative sequence

$$\mathcal{D}^-(t) = \begin{cases} 1 & \text{if } t \in \left(\alpha - \frac{T}{2}, 0\right] \text{ and } t \in \left(\alpha - T, -\frac{T}{2}\right] \\ 0 & \text{otherwise} \end{cases}$$

Consequently, the in-phase and out-of-phase components of the supply current are computed as

$$i_{\|}(t) = \frac{\mathcal{D}^+(t) + \mathcal{D}^-(t)}{2} u(t)$$

$$i_{\perp}(t) = \frac{\mathcal{D}^+(t) - \mathcal{D}^-(t)}{2} u(t) \tag{14}$$

which clearly satisfies $i(t) = i_{\|}(t) + i_{\perp}(t)$. The in-phase components can be further decomposed using (9) and (10) into an active current $i_a(t)$ and a scattered current $i_s(t)$, respectively. According to (11), the reactive current $i_r(t)$ is represented by the out-of-phase components. Based on this current decomposition, we now compute:

| $P_a$ [kW] | $D_s$ [kVA] | $Q_r$ [kVAr] | $D_h$ [kVA] | $S$ [kVA] |
|---|---|---|---|---|
| 4.397 | 9.331 | 10.315 | 0 | 14.588 |

The waveforms of the active, scattered, and reactive currents are shown in Fig. 4.

## 6. CZARNECKI'S APPROACH REVISITED

We observe that, according to the time-domain CPC approach outlined in the previous section, the TRIAC circuit actually does exhibit scattered power. This in contrast to the assertions and derivations in [1]. Let us next, to verify our results, consider the situation from the conventional CPC perspective outlined in Section 3.

From the exposition in Section 3, we readily observe that the scattered power is due to the fact that the load conductance may change with harmonic order. This occurs when the load is frequency-dependent. Considering the TRIAC circuit, the load is characterized by (13), which produces a periodic sequence of pulse wave signals. From a frequency-domain perspective, these pulses have an infinite amount of harmonic content, and can therefore be written as a Fourier series as follows:

$$\mathcal{D}_n = \frac{1}{T} \int_0^T \mathcal{D}(t) e^{-jn\omega_f t} dt$$

Since $\mathcal{D}_n$ is complex-valued, we can decompose it into

$$\mathcal{D}_n := G_n + jB_n$$

where

$$G_n = \frac{-\sin\left(\frac{3n\pi}{4}\right) + \sin(n\pi) - \sin\left(\frac{7n\pi}{4}\right) + \sin(2n\pi)}{2n\pi}$$

$$B_n = \frac{-\cos\left(\frac{3n\pi}{4}\right) + \cos(n\pi) - \cos\left(\frac{7n\pi}{4}\right) + \cos(2n\pi)}{2n\pi}$$

represent the real and imaginary parts of the load conductance and thus clearly change with harmonic order!

Proceeding along the lines of Section 3, the supply current can be expressed as

$$i(t) = \mathrm{Re}\left\{\sum_{n=-\infty}^{\infty}(G_n + jB_n)e^{jn\omega_f t}\right\}u(t)$$

which, in a similar fashion as (2), can be decomposed into

$$i_a(t) = G_e u(t).$$

$$i_s(t) = \mathrm{Re}\left\{\sum_{n=-\infty}^{\infty} G_n e^{jn\omega_f t}\right\}u(t) - G_e u(t)$$

$$i_r(t) = \mathrm{Re}\left\{\sum_{n=-\infty}^{\infty} jB_n e^{jn\omega_f t}\right\}u(t)$$

where $G_e$ represents the equivalent conductance (3). However, a drawback of the hybrid CPC approach is that in order to produce the respective powers, the results must be approximated using a finite number of harmonics, whereas the time-domain CPC method is able to produce these powers analytically. Indeed, taking 500 harmonics into account, we obtain:

| $P_a$ [kW] | $D_s$ [kVA] | $Q_r$ [kVAr] | $D_h$ [kVA] | $S$ [kVA] |
|---|---|---|---|---|
| 4.397 | 9.299 | 10.286 | 0 | 14.547 |

These values are close, but not exactly equal to the results obtained earlier. Needless to say that increasing the number of harmonics gradually improves the results, but at the expense of computational effort.

## 7. CONCLUSIONS

The results presented in [1] are critically reviewed. It is shown that the powers in a TRIAC controlled resistor circuit driven by a sinusoidal supply voltage should be decomposed into an active, reactive, and scattered component. This was motivated from both a pure CPC time-domain perspective [3] as well as the conventional hybrid CPC method [2], [4]. Harmonic power $D_h$ should be reserved for those situations in which the load actively generates harmonic power through, for example, a harmonic generating current source.

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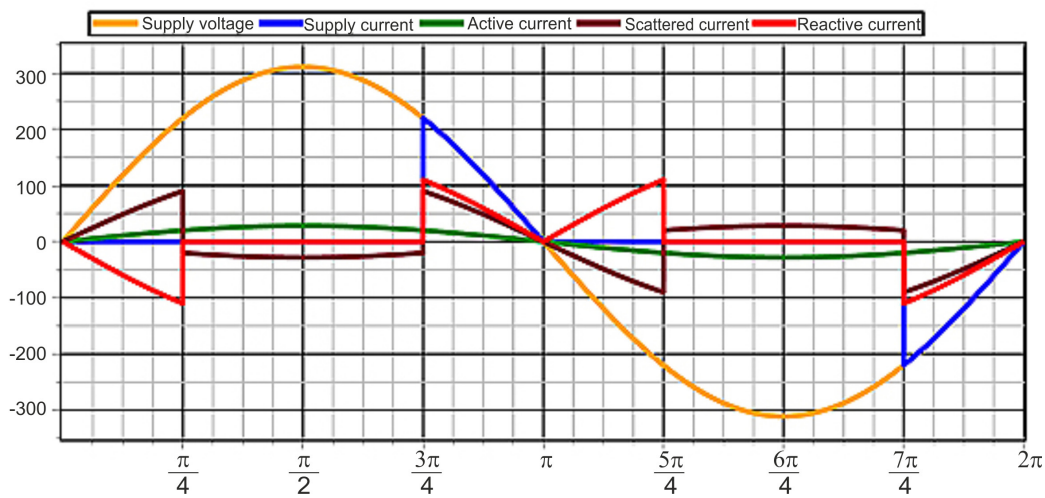


Fig. 4. Voltage and current CPC waveforms for the TRIAC circuit obtained in the time-domain

**Dimitri Jeltsema**
received the B.Eng. degree in electrical engineering from the Rotterdam University of Applied Sciences, The Netherlands, and the M.Sc. degree in systems and control engineering from the University of Hertfordshire, United Kingdom in 1996 and 2000, respectively. In May 2005 he received the Ph.D. degree with honors (cum laude) from Delft University of Technology, The Netherlands. Until 2007 he has been a post-doctoral researcher and lecturer at the Delft Center for Systems and Control. Currently he is an assistant professor at the Mathematical Physics Group of the Delft Institute of Applied Mathematics. His research includes systems and control theory of physical systems, power theory and power factor optimization, and nonlinear circuit theory.

**Jacob van der Woude**
received the M.S. degree in mathematics from the Rijksuniversiteit Groningen in 1981 and the Ph.D. degree from the Department of Mathematics and Computing Science, Eindhoven University of Technology, Eindhoven in 1987. In 1988 and 1989, he was a Research Fellow at the Center for Mathematics and Computer Science, Amsterdam. Since 1990, he is with the System Theory Group, Faculty of Electrical Engineering, Mathematics and Computer Science, Delft University of Technology, first as assistant professor and from 2000 onwards as associate professor. His current research interests include system and control theory, switching and power networks and applications of graph theory.