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Mathematical Model of DC Motor for Analysis of Commutation Processes

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***Summary*: The mathematical model of dc motor was created. In model the commutator is approximated by circuit with variable parameters. Extremely different values are assigned to circuit parameters depending on the angular position of the rotor. Model equations were solved using implicit integration method. Commutation processes of dc motor were investigated with the aid of this model. Presented model can be extended and to take into account another phenomena occurring in dc motor.**

Keywords:
mathematical model, dc motor, commutation process

1. INTRODUCTION

The methodological basis for the analysis of dc motor is the electromechanical system theory [5]. Difficulties in numerical analysis are caused by commutator. Mathematical models of commutator can be created with different degree of accuracy for representing of commutator characteristics [2, 3, 4, 6, 7]. In this paper the commutator is approximated by circuit with variable parameters. Mathematical model of dc motor with loop rotor winding has been constructed for diagnostics aims. This model creates possibility to avoid or to reduce the measuring tests as well as to perform investigations which cannot be carried out on a real object.

2. MATHEMATICAL DESCRIPTION OF COMMUTATOR DC MOTOR

The mathematical description of dc motor is derived on the following assumptions:

- the magnetic circuit is linear,
- the air-gap is uniform,
- the unipolar flux is neglected,
- the eddy currents in iron are not taken into account,
- the commutator of the motor is approximated by circuit with variable parameters.

The commutator of dc motor is approximated by resistance circuit. Extremely different values are assigned to parameters of circuits depending on the angular position of the rotor. Mathematical model of dc motor formed in this manner is a set of stiff nonlinear differential equations [1]. This model is competitive in comparison with variable structure models. Equivalent circuit of dc motor with loop rotor winding is presented in Figure 1. The commutator dc motor circuit is described by stiff differential equations:

$$\frac{d}{dt}\left(\mathbf{C}^{\mathrm{T}}\mathbf{LCi}\right)+\mathbf{C}^{\mathrm{T}}\mathbf{RCi}=\mathbf{u} \tag{1}$$

$$\mathbf{C}^{\mathrm{T}}\mathbf{LC}=\begin{bmatrix}\mathbf{L}_a & \\ & 0\end{bmatrix} \tag{2}$$

$$\mathbf{i}=\begin{bmatrix}\mathbf{i}_a \\ \mathbf{i}_b\end{bmatrix} \tag{3}$$

$$\mathbf{u}=\begin{bmatrix}\mathbf{u}_a \\ \mathbf{u}_b\end{bmatrix} \tag{4}$$

$$J\frac{d}{dt}\omega + D\omega = T_e - T_l \tag{5}$$

$$\frac{d}{dt}\varphi = \omega \tag{6}$$

$$T_e = \frac{1}{2}(\mathbf{i}_a)^{\mathrm{T}}\frac{\partial}{\partial\varphi}(\mathbf{L}_a)\ \mathbf{i}_a \tag{7}$$

The connections of the circuit elements determine the form of the **C** matrix of constraints. The introduction of the **C** matrix of constraints reduces the system of branch currents

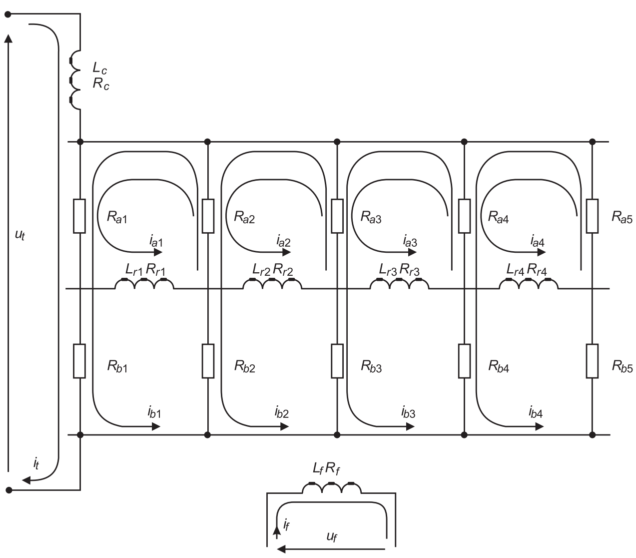

Fig. 1. Equivalent circuit of dc motor with loop rotor winding

and system of branch source voltages to systems of mesh currents and mesh source voltages:

$$\mathbf{i}_g = \mathbf{C}\begin{bmatrix} \mathbf{i}_a \\ \mathbf{i}_b \end{bmatrix} \tag{8}$$

$$\mathbf{C}^{\mathrm{T}}\mathbf{u}_g = \begin{bmatrix} \mathbf{u}_a \\ \mathbf{u}_b \end{bmatrix} \tag{9}$$

The **C** matrix of constraints is in the following form:

$$\mathbf{C} = \left[\begin{array}{cccccccc|cccccc}
1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\hline
1 & 0 & 0 & \cdots & 0 & -1 & 1 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
-1 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & -1 & 1 & 0 & \cdots & 0 & 0 \\
0 & -1 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & -1 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 1 \\
0 & 0 & 0 & \cdots & -1 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1 \\
\hline
0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & -1 & 1 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & -1 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & -1 & 1 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & -1
\end{array}\right] \tag{10}$$

The $\mathbf{i}_a$, $\mathbf{i}_b$, $\mathbf{u}_a$, $\mathbf{u}_b$ vectors and $\mathbf{L}$, $\mathbf{L}_a$, $\mathbf{R}$ matrices for considered machine are of the following forms:

$$\mathbf{i}_a = \begin{bmatrix} i_{a1} \\ i_{a2} \\ i_{a3} \\ \vdots \\ i_{a(K-1)} \\ i_{aK} \\ i_t \\ i_f \end{bmatrix} \tag{11}$$

$$\mathbf{i}_b = \begin{bmatrix} i_{b1} \\ i_{b2} \\ i_{b3} \\ \vdots \\ i_{b(K-2)} \\ i_{b(K-1)} \end{bmatrix} \tag{12}$$

$$\mathbf{u}_a = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ u_t \\ u_f \end{bmatrix} \tag{13}$$

$$\mathbf{u}_b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} \tag{14}$$

$$\mathbf{L} = \begin{bmatrix}
L_{r1} & . & . & M_{1K} & M_{1c} & M_{1f} & 0 & . & . & 0 \\
. & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . \\
M_{K1} & . & . & L_{rK} & M_{Kc} & M_{Kf} & 0 & . & . & 0 \\
M_{c1} & . & . & M_{cK} & L_c & M_{cf} & 0 & . & . & 0 \\
M_{f1} & . & . & M_{fK} & M_{fc} & L_f & 0 & . & . & 0 \\
0 & . & . & 0 & 0 & 0 & 0 & . & . & 0 \\
. & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . \\
0 & . & . & 0 & 0 & 0 & 0 & . & . & 0
\end{bmatrix} \tag{15}$$

$$\mathbf{L}_a = \begin{bmatrix}
L_{r1} & M_{12} & \cdots & M_{1K} & M_{1c} & M_{1f} \\
M_{21} & L_{r2} & \cdots & M_{2K} & M_{2c} & M_{2f} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
M_{K1} & M_{K2} & \cdots & L_{rK} & M_{Kc} & M_{Kf} \\
M_{c1} & M_{c2} & \cdots & M_{cK} & L_c & M_{cf} \\
M_{f1} & M_{f2} & \cdots & M_{fK} & M_{fc} & L_f
\end{bmatrix} \tag{16}$$

$$\mathbf{R} = \mathrm{diag}\,(R_{r1}, \ldots, R_{rK}, R_c, R_f, R_{a1}, \ldots, R_{aK}, R_{b1}, \ldots, R_{bK}) \tag{17}$$

The inductances depend on the rotor position and are obtained by finite element calculations and then expressed in Fourier series [8]. A smaller accuracy is achieved when the inductances are calculated by means of the air-gap permeance function:

$$\lambda = \frac{\mu_o}{\delta'}\left[1+\sum_{\nu=1}^{N} a_\nu \cos(\nu y)\right]\left[1+\sum_{\rho=1}^{N} b_\rho \cos\left(\rho\left(y-\varphi+\frac{\mathrm{tg}\gamma}{r}z\right)\right)\right] \tag{18}$$

$$\delta' = k_C k_{Fe} \delta \tag{19}$$

Resistances R_{am}, R_{bm}, $m = 1,.., K$ depend on the rotor position. These resistances in the motor with rotor loop winding are determined by the formulae:

$$R_{am}(t) = h\left(\left(\varphi(t)+\varphi_3+(m-1)\ \varepsilon+\frac{\pi}{2p}\right)\left(mod\frac{2\pi}{p}\right)-\varphi_3\right) \quad (20)$$

$$R_{bm}(t) = h\left(\left(\varphi(t)+\varphi_3+(m-1)\ \varepsilon+\frac{3\pi}{2p}\right)\left(mod\frac{2\pi}{p}\right)-\varphi_3\right) \quad (21)$$

for $m \in \{1, 2, ..., K\}$

$$\varepsilon = \frac{2\pi}{K}\ ,\ \ \varphi_3 = 0.5\left(\frac{b}{r_c}+\frac{2\pi}{K}\right) \quad (22)$$

A function $f(\varphi')$ is in the form:

$$h(\varphi') = min\left\{max\left\{\left(\frac{R_z-R_p}{\varphi_2-\varphi_1}\right)\left(\left|\ \varphi'\right|-\varphi_1\right)\ ,R_p\right\}\ ,R_z\right\} \quad (23)$$

where:
$|\varphi'|$ — absolute value of function,
$\varphi_1 = \eta_1\varphi_3$, $\varphi_2 = \eta_2\varphi_3$, η_1, η_2 — coefficients.
In the model equations and figure the notation was applied:
K — number of commutator bars,
C — matrix of constraints,
L — matrix of branch inductances,
$\mathbf{L}_a$ — matrix of mesh inductances,
R — matrix of branch resistances,
$\mathbf{i}_g$ — vector of branch currents,
i — vector of mesh currents,
$\mathbf{i}_a$ — resistance-inductance mesh currents vector,
$\mathbf{i}_b$ — resistance mesh currents vector,
$\mathbf{u}_g$ — vector of branch source voltages,
u — vector of mesh source voltages,
$\mathbf{u}_a$ — resistance-inductance mesh source voltages vector,
$\mathbf{u}_b$ — resistance mesh source voltages vector,
T_e, T_l — electromagnetic and load torques,
λ — air-gap permeance,
μ_o — magnetic permeability of the air,
r — mean radius of the air-gap,
δ, δ' — value and equivalent value of the air-gap,
k_C — Carter coefficient,
k_{Fe} — iron saturation coefficient,
L_{r1} ,..., L_{rK} — self-inductances of rotor coils,
L_c — self-inductance of commutation winding,
L_f — self-inductance of field winding,
M_{ij}, $i = 1,\dots, K+2, j=1,\dots,K+2, i \neq j$ — mutual inductances,
R_{r1} ,..., R_{rK} — resistance of rotor coils,
R_c — resistance of commutation winding,
R_f — resistance of field winding,
R_{a1} ,..., R_{aK}, R_{b1} ,..., R_{bK} — commutator resistances,
R_p — min. value of brush-coil contact resistance,
R_z — max. value of brush-coil contact resistance,
b — width of brush,
r_c — radius of commutator,
J — inertia moment,
D — damping coefficient.

3. SIMULATION RESULTS

The effective implicit integration method is used for solution of stiff equations of model. Location algorithm which allows to determine the parameter value change instants with required accuracy is employed. Simulation parameters are determined by the transient state duration and required accuracy of calculations. The numerical investigations were carried out using FORTRAN language. The design data and values of the parameters of motor were: P_N = 13 kW, U_N = 75 V, I_N = 200 A, U_{fN} = 220 V, I_{fN} = 4 A, n_N = 700 obr/min, p = 2, r = 0.15 m, r_c = 0.130 m, b = 0.025 m, δ_c = 0.006 m, δ_f = 0.004 m, R_c = 0.0064 Ω, R_f = 50 Ω, R_{r1} = R_{r2} = ... = R_{rK} = 0.00093 Ω, K = 126, R_p = 0.01 Ω, R_z = 10000 Ω, J = 0.9 kgm^2, D = 0 Nms/rad. Operation of this motor was simulated at constant armature voltage U_t = 75 V, constant field voltage U_f = 220 V, load torque T_l = 20 Nm, initial value of field current $i_f(0)$ = 3.2 A and initial value of velocity of rotor $\omega(0)$ = 70.16 rad/s. Selected calculation results are presented in Figures 2–6.

4. CONCLUSIONS

Mathematical model of commutator dc motor with loop and other type winding of rotor is a set of stiff nonlinear differential equations. This mathematical model enables to investigate the commutation processes of dc motor.

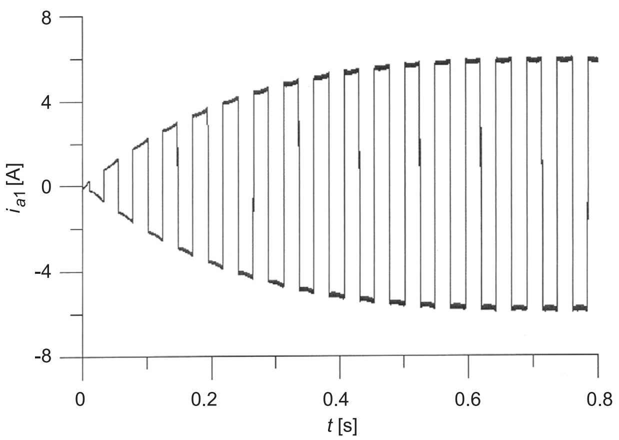

Fig. 2. Current of one coil of rotor

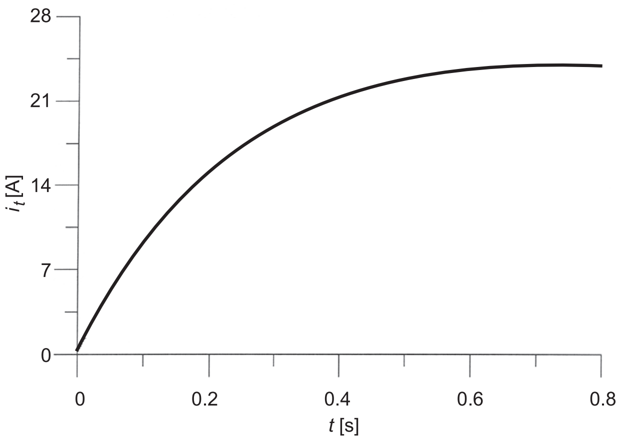

Fig. 3. Current of armature winding

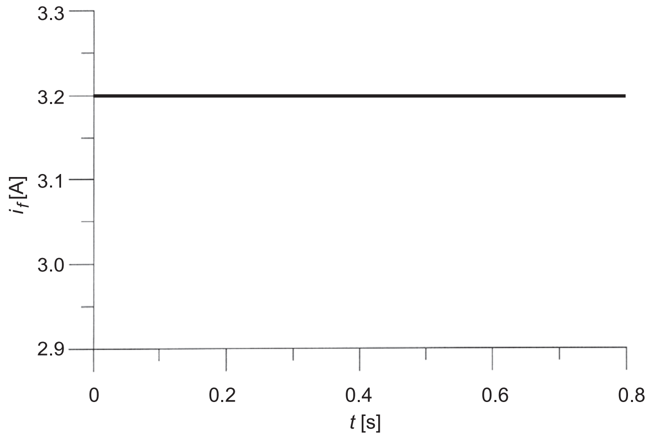

Fig. 4. Current of field winding

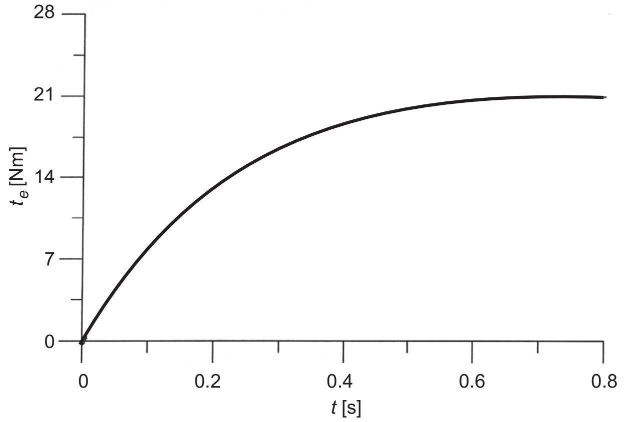

Fig. 5. Electromagnetic torque

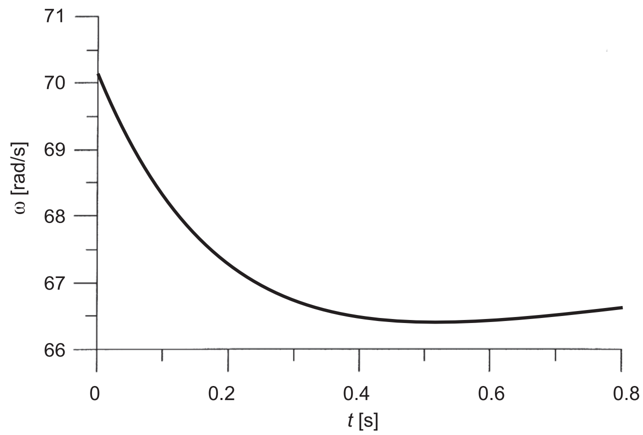

Fig. 6. Angular velocity of rotor

Considered model can be extended and to take into account another phenomena occurring in dc motor (e.g. saturation of magnetic circuit, influence of temperature on brush-coil resistance, equalizing connections). Such model can be used in investigation of semiconductor converter fed commutator motor. The rotor coil currents, armature current, field current, electromagnetic torque and angular velocity are depending on parameters of dc motor. Effects of commutation processes are visible in considered waveforms. The differences between calculations and measurements results are caused by still inaccurate description of physical phenomena in commutator dc motor.

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