Electrical Power Quality and Utilisation, Journal Vol. XII, No. 2, 2006

Extreme Possibilities of Circuital Models of Electric Machines

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***Summary*: The paper presents possibilities of so-called circuital mathematical models of electric machines, in which the machine is treated as a set of magnetically coupled coils. It is a very popular class of models used for analysis of operational conditions of electric machines. However, the progress in numerical solutions of electromagnetic filed causes the interest of researchers to shift into so-called field-models. This paper shows the author's opinion on extreme possibilities of circuital models putting main attention to their creation, simplification and solving.**

Key words:
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1. INTRODUCTION

Circuital models are the most popular research tools of electric machines. Such models are commonly used for investigation of electric machine properties itself, as well as for solving operation problems of more complex systems, in which electric machine is an important element. Usefulness of such models is out of the question, but nowadays field models basing on numerical solving of field equations, become concurrent. The field models have a great advantage for electric machine design when a shape and dimensions of a machine electromagnetic circuit have to be found. However, for prediction of machine properties the advantage of field models is not so evident.

A very fast development of computer hardware and software for electromagnetic, mechanic and thermal fields causes the field models to be used for prediction of electric machines technical data as well as for modeling operational problems related to electric machines in drive or in power system. Generally, the field models have to be solved numerically, so there is no chance to predict qualitatively machine properties by such models. Field models are time consuming and need powerful computers and modern software, so are rather expensive.

Circuital models treat an electric machine as a set of magnetically coupled coils located on a stator and rotor and an elementary mechanical system with constant parameters. Such models ensure rather a simple dimensionless description of electric, magnetic and mechanical phenomena in electric machines. All of them have to be considered together to model electromechanical energy conversion processes in electrical machines.

It is commonly accepted that the Lagrange's formalism, which is based on the co-energy function of the whole system, is the most effective way to model a process of energy conversion. However, it requires an assumption that the co-energy stored a machine magnetic circuit is a unique function of currents and a mechanical angle. Interpreting this assumption physically means that the currents paths have to be well determined and characteristics of magnetic material have to be unique, too. Making above assumptions has strong physical consequences—conducting continuous areas and ferromagnetic materials with a hysteresis loop is not accepted theoretically in such models. Usually, conducting areas are discretized enlarging a number of currents.

The co-energy function is determined by characteristics of coils creating an electric machine. The coil characteristics determine the coils linked fluxes as function of all currents $i_1,..., i_N$ and a mechanical angle φ of a stator and rotor mutual position:

$$\psi_n = \psi_n(i_1,...,i_N,\varphi) \quad \text{for} \quad n \in \{1,...,N) \tag{1.1}$$

The co-energy function is given by the formula:

$$E_{om}(\mathbf{i},\varphi) = \frac{1}{2}J\omega^2 + \sum_{n=1}^{N}\int_0^{i_n}\psi_n(i_1,...,i'_n,0,...,0,\varphi)\mathrm{d}i'_n \tag{1.2}$$

Leaving detailed considerations well known in literature, the equations of circuital models of electric machines take the general form:

$$\frac{\mathrm{d}}{\mathrm{dt}}\Psi(\mathbf{i},\varphi) + \mathbf{R}\ \mathbf{i} = \mathbf{u} \tag{1.3a}$$

$$J\frac{\mathrm{d}^2\varphi}{\mathrm{dt}^2} = \frac{\partial E_{om}(\mathbf{i},\varphi)}{\partial\varphi} + T_m \tag{1.3b}$$

where $\Psi(\mathbf{i}, \varphi)$, $\mathbf{i}(t)$, $\mathbf{u}(t)$ are vectors of linked fluxes, currents and voltages respectively and $\mathbf{R}$ is a square resistance matrix.

This paper presents advantages and limitations of circuital approach to modelling of electric machines and focuses on extreme possibilities of such models.

2. LINEAR CIRCUITAL MODELS

Circuital models obtained under the assumption on linearity of machine magnetic circuit are especially attractive due to their simplicity. Because of that they become a base for modelling electromagnetic phenomena in electric machines. Linearity of magnetic circuit leads to linear relationships between linked fluxes and currents for all coils, which is mathematically expressed by the formula:

$$\Psi(\mathbf{i},\varphi) = \mathbf{L}(\varphi)\,\mathbf{i} \tag{2.1}$$

and the co-energy function is given by the quadratic form of currents:

$$E_{om}(\mathbf{i},\varphi) = \tfrac{1}{2}\mathbf{i}^T\mathbf{L}(\varphi)\,\mathbf{i} \tag{2.2}$$

Equations of machine models take the form:

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\mathbf{L}(\varphi)\,\mathbf{i}\right) + \mathbf{R}\;\mathbf{i} = \mathbf{u} \tag{2.3a}$$

$$\mathrm{J}\frac{\mathrm{d}^2\varphi}{\mathrm{dt}^2} = \tfrac{1}{2}\mathbf{i}^T\,\frac{\partial\mathbf{L}(\varphi)}{\partial\varphi}\,\mathbf{i} + T_m \tag{2.3b}$$

in which the most important is the inductance matrix $\mathbf{L}(\varphi)$. Self and mutual inductances of all coils, depending on design data of machine windings and geometry of machine magnetic circuit, substitute it. The inductance matrix is symmetric ($\mathbf{L}(\varphi) = \mathbf{L}^T(\varphi)$), positively determined (det $\mathbf{L}(\varphi) > 0$) and periodic with respect to the angle φ ($\mathbf{L}(\varphi) = \mathbf{L}(\varphi + 2\pi)$). Each of these mathematical properties has a physical background: independence of stored energy from the way it has been accumulated, no ideal magnetic coupling of coils representing machine windings and periodicity of phenomena in the machine with respect to the rotor position angle.

Determination of all inductances as functions of the angle φ is an essential problem. Usually two groups of coils representing stator and rotor windings are distinguished and the inductance matrix is written in the form:

$$\mathbf{L}(\varphi) = \begin{bmatrix} \mathbf{L}^{\sigma}{}_{\mathrm{ss}} + \mathbf{L}^{\mathrm{m}}{}_{\mathrm{ss}}(\varphi) & \left(\mathbf{L}^{\mathrm{m}}{}_{\mathrm{rs}}(\varphi)\right)^T \\ \mathbf{L}^{\mathrm{m}}{}_{\mathrm{rs}}(\varphi) & \mathbf{L}^{\sigma}{}_{\mathrm{rr}} + \mathbf{L}^{\mathrm{m}}{}_{\mathrm{rr}}(\varphi) \end{bmatrix} \tag{2.4}$$

where:

$\mathbf{L}^{\sigma}{}_{\mathrm{ss}}$ — is a leakage inductances matrix;
$\mathbf{L}^{\mathrm{m}}{}_{\mathrm{ss}}(\varphi)$ — is the main inductances matrix of stator coils;
$\mathbf{L}^{\sigma}{}_{\mathrm{rr}}$ — is a leakage inductances matrix;
$\mathbf{L}^{\mathrm{m}}{}_{\mathrm{rr}}(\varphi)$ — is the main inductances matrix of rotor coils;
$\mathbf{L}^{\mathrm{m}}{}_{\mathrm{rs}}(\varphi)$ — is a mutual inductances matrix of stator and rotor coils.

By means of this methodology it is possible to create a very detailed model, in which each individual coil, both on a stator and a rotor sides, is treated as an independent one. Let such a set of coils be located in a cylindrical magnetic circuit with a non-smooth air gap both on a stator and a rotor sides. Such a model can be considered a base model and any individual model for a given machine can be obtained from it. It is relatively easy to find the inductance matrix for such a basic model. Let M arbitrary coils, which have magnetic axes on angular positions $\alpha_1, \alpha_2, ..., \alpha_{\mathrm{M}}$ with respect to a stator reference axes, be located on the stator and N arbitrary coils, which have magnetic axes on angular positions $\beta_1, \beta_2, ..., \beta_{\mathrm{N}}$ with respect to a rotor reference axes, be located on the rotor side. Let each coil produce the MMF with full Fourier spectrum. In such a case all self and mutual inductances depend on three variables and are periodic with respect to each of them [11] and can be expanded onto triples Fourier series. For self and mutual inductances of stator coils these are: a location angle of magnetic axes of individual stator coil α_m, an angle between magnetic axes of two considered stator coils $\alpha_m - \alpha_k$ and a rotor position angle φ. For self and mutual inductances of rotor coils as these three variables can be chosen: a location angle of magnetic axes of individual rotor coil with respect to stator reference frame $\varphi + \beta_n$, an angle between magnetic axes of two considered rotor coils $\beta_n - \beta_k$ and the rotor position angle φ. For mutual inductances between stator and rotor coils the following three variables can be used: a location angle of magnetic axes of individual rotor coil with respect to a stator reference frame $\varphi + \beta_n$, an angle between magnetic axes of two considered rotor coils one on a stator and one on a rotor $\varphi + \beta_n - \alpha_m$ and a rotor position angle φ. The general forms of inductance matrices $\mathbf{L}^{\mathrm{m}}{}_{\mathrm{ss}}(\varphi)$, $\mathbf{L}^{\mathrm{m}}{}_{\mathrm{rr}}(\varphi)$ and $\mathbf{L}^{\mathrm{m}}{}_{\mathrm{rs}}(\varphi)$ are given in Appendix 1. In all above mentioned matrices there appear triples sums in which summing up index v depends on harmonics orders of coils MMFs. Indexes r, s depend on harmonic orders of a permeance function representing geometry of air-gap zone, which, for a non-smooth air gap on stator and rotor sides, can be written in the form of double Fourier series:

$$\lambda(x,\varphi) = \sum_r \sum_s \Lambda_{r,s}\mathrm{e}^{\mathrm{j}rx}\mathrm{e}^{\mathrm{j}s\varphi} \tag{2.5}$$

Leakage inductance matrices $\mathbf{L}^{\sigma}{}_{\mathrm{ss}}$ and $\mathbf{L}^{\sigma}{}_{\mathrm{rr}}$ can be also very carefully analysed analyzed accounting for the placement of individual coils in the slots and in the coil outhangs.

This base model is very detailed and rather complicated but it just shows the possibilities of linear circuital models of electric machines. It requires a lot of parameters, mainly inductances. Usually, the inductances for linear models are calculated from a very simple analysis of magnetic field distribution in the air gap. It is one-dimensional analysis, in which a radial component of flux density in the air gap is determined as a function of angular position along circumference. There are many formulas in literature determining the self and mutual inductances of an arbitrary pair of coils located in the air gap zone [2, 9, 10, 11]. Coils MMF functions and the permenace function of the air gap are needed for that. For example, in [11] such a general formula is given, but its presentation and interpretation would be too long. The inductance for a linear base model could be calculated by numerical solution of field equations in a machine magnetic circuit, assuming linearity of magnetic material. Nowadays computers and magnetic field solvers allow doing it very fast and effectively. While computing inductances it should be kept in mind that in any electric machine a set of coils is specially arranged, which leads to a specific structure of inductance matrices. It can happen that a distortion of the matrix structure can lead to more serious mistakes than a little bit less precise calculation of inductances with the matrix structure.

In order to explain this statement, let us consider the symmetrically located sets of identical coils on a stator and analogously on a rotor. For such a case it is enough to know

three functions. The matrix $\mathbf{L}^{m}_{ss}(\varphi)$ is determined by one function of three variables mentioned above $L_s(\alpha_m - \alpha_k, \alpha_m, \varphi)$, for the matrix $\mathbf{L}^{m}_{rr}(\varphi)$ it is enough to know the function $L_r(\beta_n - \beta_k, \varphi + \beta_n, \varphi)$ and the matrix $\mathbf{L}^{m}_{rs}(\varphi)$ is determined by the function $L_{rs}(\varphi + \beta_n - \alpha_m, \varphi + \beta_n, \varphi)$. As it has already been written, all these functions are periodic with respect to each of their variables. The structures of these matrices are given in Appendix 2.

From a base model (operating with individual coils in a cylindrical magnetic circuit with a non-smooth air gap on stator and rotor sides) models of any individual machine having some specific features of magnetic circuit and coils arrangement can be obtained. For a magnetic circuit with a non-smooth air gap on a stator or a rotor side inductances become functions of two variables and for a smooth air-gap inductances they depend on one variable only – an angle between magnetic axes of coils. The numbers of summings up in the matrices shown in Appendix 1 are reduced respectively. In an electric machine the elementary coils are connected in multi-turn coils located in one slot, which are then grouped, and the groups constitute phases. The way of coils connections is mathematically expressed by a Kron constrain matrix. By means of Kron constrain matrices every internal asymmetry in a machine, both designed as well arisen a result of a fault can be mathematically expressed. Denoting Kron constrain matrices of stator and rotor coils by $\mathbf{C}_s$ and $\mathbf{C}_r$ respectively, the inductance matrix of a machine takes the form:

$$\mathbf{L}_C(\varphi) = \begin{bmatrix} \mathbf{C}_s^T & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_r^T \end{bmatrix} \begin{bmatrix} \mathbf{L}^{\sigma}_{ss} + \mathbf{L}^{m}_{ss}(\varphi) & \left(\mathbf{L}^{m}_{rs}(\varphi)\right)^T \\ \mathbf{L}^{m}_{rs}(\varphi) & \mathbf{L}^{\sigma}_{rr} + \mathbf{L}^{m}_{rr}(\varphi) \end{bmatrix} \begin{bmatrix} \mathbf{C}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_r \end{bmatrix} \tag{2.6}$$

Introducing Kron matrices a machine is described by independent currents, so the mathematical description of a machine is unique. However, a unified approach starting from a base model cannot be the fastest way to a machine model for each case. Because linear models can be created very easily, an individual model should use, depending on requirements, directly group of coils or a whole phase, instead of elementary coils.

Any model has to be effectively solved, which is a separate problem for electric machine models. Commercial software packages like MatLab, MatCad, Mathematica and others are very well prepared for solving high order differential equations, even with complex coefficients, which in fact winds up the problem. Thus, equations created above can be numerically solved together with equations of co-operating systems: mechanical, power electronics, etc. However, they are rather complicated and any of their simplification is profitable and leads to reduction of computing time, does not require a very advanced hardware and reduces the time necessary for preparation and performance of research. Simplification of electrical model equations is important from the cognitive point of view, facilitating physical interpretation of phenomena inside a machine and properties of model solutions.

An electric machine at circuital approach is treated as a set of coils, so for simplification of its model equations, a system analysis methods can be applied. The most popular and a very effective way is a transformation of variables, which leads to decomposition of machine equations into independent sub-systems, decreasing order of equations, which have to be solved together. Application of symmetrical components of currents and voltages both on stator as well as rotor sides, which are defined by linear transformation with a matrix of the general form, is especially effective:

$$\mathbf{S} = \frac{1}{\sqrt{K}} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & c & c^2 & \cdots & c^{(K-1)} \\ 1 & c^2 & c^4 & \cdots & c^{2(K-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & c^{(K-1)} & c^{2(K-1)} & \cdots & c^{(K-1)(K-1)} \end{bmatrix}, \quad c = e^{j\frac{2\pi}{K}} \tag{2.7}$$

The inductance matrix transformed into symmetrical components is Hermitian and takes the form:

$$\mathbf{L}_S(\varphi) = \begin{bmatrix} \mathbf{S}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_r \end{bmatrix} \begin{bmatrix} \mathbf{C}_s^T & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_r^T \end{bmatrix} \times$$

$$\times \begin{bmatrix} \mathbf{L}^{\sigma}_{ss} + \mathbf{L}^{m}_{ss}(\varphi) & \left(\mathbf{L}^{m}_{rs}(\varphi)\right)^T \\ \mathbf{L}^{m}_{rs}(\varphi) & \mathbf{L}^{\sigma}_{rr} + \mathbf{L}^{m}_{rr}(\varphi) \end{bmatrix} \times \tag{2.8}$$

$$\times \begin{bmatrix} \mathbf{C}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_r \end{bmatrix} \begin{bmatrix} \mathbf{S}_s^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{S}_r^{-1} \end{bmatrix}$$

Description by symmetrical components puts in order machine equations and every symmetry of magnetic circuit and of coils arrangement reflects on their structure so description by symmetrical components is specially effective for fully symmetrical machines.

An important simplification of machine equation can be obtained using the other transformation, which eliminates dependence of the inductance matrix $\mathbf{L}_s(\varphi)$ on the rotor position angle. Matrices of such transformations depend on that angle and they turn the previously defined symmetrical components on the complex plain. Unfortunately, not for every case these matrixes can be found. Leaving the details, general conditions for the structure of the inductance matrix $\mathbf{L}_s(\varphi)$ allowing to eliminate its angle dependence are given below. Firstly, the $\mathbf{L}_s(\varphi)$ matrix should be approximated by the Fourier series with a limited number of terms:

$$\mathbf{L}_S(\varphi) = \sum_{k=-K}^{K} \mathbf{L}_k e^{jk\varphi} \tag{2.9}$$

choosing harmonic orders so as to have at most one term $L_m e^{jm\varphi}$ of this series in any matrix element. These transformation matrices have the general form:

$$\Lambda_s(\varphi) = \mathrm{diag}\left[e^{jk_{s,1}\varphi} \quad \cdots \quad e^{jk_{s,M}\varphi} \right] \tag{2.10a}$$

$$\Lambda_r(\varphi) = \mathrm{diag}\left[e^{jk_{r,1}\varphi} \quad \cdots \quad e^{jk_{r,N}\varphi} \right] \tag{2.10b}$$

where M and N are respective dimensions of stator and rotor quantities. Secondly, the equations (2.11) should be checked. When an element of the matrix $\mathbf{L}_s(\varphi)$ does not contain any term $L_m e^{jm\varphi}$ a respective equation should be removed from the equations (2.11). If these equations can be fulfilled, a set of numbers $\{k_{s,1},...,k_{s,M}\}$ and $\{k_{r,1},...,k_{r,n}\}$ is not unique, if not the Fourier series (2.9) the number of considered harmonic should be limited or the orders of harmonic taken into account in the inductance matrix should be changed.

$$\begin{bmatrix} \begin{bmatrix} m_{1,2} \\ m_{1,3} \\ \vdots \\ m_{1,M} \\ m_{1,M+1} \\ m_{1,M+2} \\ \vdots \\ m_{1,M+N} \end{bmatrix} \\ \begin{bmatrix} m_{2,3} \\ m_{2,4} \\ \vdots \\ m_{2,M} \\ m_{2,M+1} \\ m_{2,M+2} \\ \vdots \\ m_{2,M+N} \end{bmatrix} \\ \vdots \\ \begin{bmatrix} m_{M+N-1,N+M} \end{bmatrix} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 & \cdots & \cdots & 0 \\ -1 & 0 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & & \vdots \\ -1 & 0 & 0 & 0 & \cdots & \cdots & 1 \end{bmatrix} \\ \begin{bmatrix} 0 & -1 & 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & -1 & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \\ \vdots \\ \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & -1 & 1 \end{bmatrix} \end{bmatrix} \begin{bmatrix} \begin{bmatrix} k_{s,1} \\ k_{s,2} \\ \vdots \\ k_{s,M} \end{bmatrix} \\ \begin{bmatrix} k_{r,1} \\ k_{r,2} \\ \vdots \\ k_{r,N} \end{bmatrix} \end{bmatrix} \tag{2.11}$$

As a result of transformations of matrices depending on the angle (2.10) a new inductance matrix, Hermitian also, but with constant elements, is obtained:

$$\mathbf{L_N} = \begin{bmatrix} \Lambda_s(\varphi)\cdot\mathbf{S}_s & \mathbf{0} \\ \mathbf{0} & \Lambda_r(\varphi)\cdot\mathbf{S}_r \end{bmatrix} \begin{bmatrix} \mathbf{C}_s^T & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_r^T \end{bmatrix} \cdot$$

$$\cdot \begin{bmatrix} \mathbf{L}^{\sigma}{}_{ss} + \mathbf{L}^{m}{}_{ss}(\varphi) & \left(\mathbf{L}^{m}{}_{rs}(\varphi)\right)^T \\ \mathbf{L}^{m}{}_{rs}(\varphi) & \mathbf{L}^{\sigma}{}_{rr} + \mathbf{L}^{m}{}_{rr}(\varphi) \end{bmatrix} \cdot \tag{2.12}$$

$$\cdot \begin{bmatrix} \mathbf{C}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_r \end{bmatrix} \begin{bmatrix} (\Lambda_s(\varphi)\cdot\mathbf{S}_s)^{-1} & \mathbf{0} \\ \mathbf{0} & (\Lambda_s(\varphi)\cdot\mathbf{S}_r)^{-1} \end{bmatrix}$$

and model equations take the form:

$$\mathbf{L_N}\frac{d\mathbf{i_N}}{dt} - j\,(\mathbf{K}\,\mathbf{L_N})\,\mathbf{i_N} + \mathbf{R_N}\,\mathbf{i_N} = \mathbf{u_N} \tag{2.13a}$$

$$J\frac{d^2\varphi}{dt^2} = j\,(\overset{*}{\mathbf{i_N}})^T\left(\mathbf{L_N K} - \mathbf{K L_N}\right)\,\mathbf{i_N} + T_m \tag{2.13b}$$

in which by **K** a diagonal matrix of the form is denoted:

$$\mathbf{K} = \mathrm{diag}\left[k_{s,1} \quad \cdots \quad k_{s,M} \quad k_{r,1} \quad \cdots \quad k_{r,N} \right] \tag{2.14}$$

The equations in the form (2.13) are effortless for numerical integration and essentially simplify an analysis of steady state at a constant angular velocity.

3. MODELS ACCOUNTING FOR MAGNETIC NON-LINEARITY

Non-linearity of magnetic circuit highly complicates the electric machine equations. The equations (1.3) seem to be simple, but in fact they are rather complicated because of non-linear coil characteristics, i.e. relations between linked fluxes and currents. A set of functions describing those characteristics:

$$\psi_n(i_1,...,i_N,\varphi), \quad \text{for} \quad n = \{1,...,N\} \tag{3.1}$$

has to insure uniqueness of the co-energy function, so the functions (3.1) have to fulfill the conditions:

$$\frac{\partial\psi_n}{\partial i_k} = \frac{\partial\psi_k}{\partial i_n} \quad \text{for} \quad n,k = \{1,...,N\} \tag{3.2}$$

For sets having numerous coils a determination and adjustment of characteristic description are very difficult problems, practically unsolved till today. It is very difficult to find relations between characteristics like those for inductances in the matrices (2.6). Nowadays it is possible to describe analytically the characteristics for a set of two or three coils. Undoubtedly, an analytical description of characteristics for a set of non-linear coils is an open problem, which has to be solved for a further development of electric machines circuital models.

An alternative is determination of the co-energy as a function of all currents and a rotor position angle $E_{om}(i_1,..., i_N, \varphi)$, from which the coil characteristics can be found:

$$\psi_n(i_1,...,i_N,\varphi) = \frac{\partial E_{om}(i_1,...,i_N,\varphi)}{\partial i_n} \tag{3.4}$$

Remembering that the values of co-energy can be calculated numerically only, by solving magnetic field equations, the co-energy function can be only approximated. When a number of independent variables (mainly currents) is large the co-energy function can be approximated by a

multivariable power series, which requires a lot of coefficients. This method, quite evident, has not been practically used till now in circuital models yet.

However, this approach can be simplified by looking for qualitative properties of the co-energy function, keeping in mind that the state of electric machine magnetic circuit depends on all coils. A heuristic determination of the co-energy as a function of the total MMFs of all machine windings is very effective. It is possible when leakage fluxes are considered separately, which is a strong assumption, but gives a chance to introduce magnetic non-linearity of the main magnetic circuit into a circuital model by not a very complicated way. Thus, in the co-energy function two terms are indicated: one representing the co-energy in leakage zones and the other representing the co-energy in the main magnetic circuit:

$$E_{om}(i_1,...,i_N,\varphi) = E_{o\sigma}(i_1,...,i_N) + E_{og}(i_1,...,i_N,\varphi) \quad (3.5)$$

Magnetic circuits of leakage fluxes are very often locally separated zones and their magnetic state is determined by one or two coil currents. To reduce a list of variables of the second term in (3.5) "harmonic equivalent magnetizing currents" [11], which in fact represent a sum of MMFs of all machine windings for individual space harmonic, can be introduced. A very spectacular effect can be obtained considering the main space harmonic only. In this case co-energy depends on the parameters of the total MMF of this harmonic, which are its magnitude and angular position along circumference. The MMF magnitude is represented by an equivalent magnetic current, which depends on positions and currents of all machine windings and also on a rotor position. Considering higher space harmonic, each of them add a new pair of variables: a respective equivalent current and an angle. Remarks mentioned above allow to change a list of the other term of the co-energy function $E_{og}(i_1,..., i_N, \varphi)$:

$$E_{og}(i_1,...,i_N,\varphi) = E_{og}(\cdots, i_{\mu,k}, k\eta_k, i_{\mu,l}, l\eta_l, \cdots) \quad (3.6)$$

where {...,k, l,...} denote a set of harmonic orders taken into account. In practice, only MMF space harmonic with highest amplitudes decide on saturation of the main magnetic circuit, so a number of variables of the co-energy function is very limited.

The total MMF of all machine windings, expressed by equivalent harmonic magnetizing currents $i_{\mu,\rho}$ and respective angles $\rho\eta_\rho$ has the form:

$$\Theta(x,t) = \frac{2}{\pi}\sum_{\rho=1}^{\infty}\frac{i_{\mu,\rho}(t)}{\rho}\cos\rho\left(x-\eta_\rho(t)\right) \quad (3.7)$$

A square of the equivalent magnetizing current of ρ harmonic is given by a quadratic form:

$$\left(i_{\mu,\rho}\right)^2 = \sum_{n=1}^{N}\sum_{k=1}^{N}(w_n k_{n,\rho}\ i_n)\cdot(w_k k_{k,\rho}\ i_k)\cdot\cos\rho(\alpha_n-\alpha_k) \quad (3.8)$$

where:

w_n — is a coil turn number,

$k_{n,\rho}$ — is a winding factor,

$a_1,..., a_N$ — are the position angles of magnetic axes of all windings.

The angle $\rho\eta_\rho$ is determined by the formula:

$$\tan\left(\rho\eta_\rho\right) = \frac{\sum_{n=1}^{N}(w_n k_{n,\rho}\ i_n)\cdot\sin(\rho\ \alpha_n)}{\sum_{n=1}^{N}(w_n k_{n,\rho}\ i_n)\cdot\cos(\rho\ \alpha_n)} \quad (3.9)$$

Introducing new variables into the co-energy function, the Lagrange equations of electrical machines with a non-linear magnetic circuit take the form:

$$\frac{d}{dt}\left(\frac{\partial E_{o\sigma}}{\partial i_n}\right) + \frac{d}{dt}\sum_{\rho}\left(\frac{\partial E_{og}}{\partial i_{\mu,\rho}}\frac{\partial i_{\mu,\rho}}{\partial i_n} + \frac{\partial E_{og}}{\partial(\rho\eta_\rho)}\frac{\partial(\rho\eta_\rho)}{\partial i_n}\right) = u_n - R_n\ i_n \quad (3.10a)$$

$$\text{for} \quad n \in \{1,...,N\}$$

$$J\frac{d^2\varphi}{dt^2} = \sum_{\rho}\left(\frac{\partial E_{og}}{\partial i_{\mu,\rho}}\frac{\partial i_{\mu,\rho}}{\partial\varphi} + \frac{\partial E_{og}}{\partial(\rho\eta_\rho)}\frac{\partial(\rho\eta_\rho)}{\partial\varphi}\right) + T_m \quad (3.10b)$$

Appendix 2 presents their detailed structure. It should be noticed that the matrices in these equations follow from localization of windings along circumference of the air gap, so the application of symmetrical components can simplify the matrix structure, analogously to linear models.

In simpler cases qualitative properties of the co-energy function E_{og} follow from engineering intuition. For an electric machine with mono-harmonic windings of 'p' order and a smooth air gap the co-energy function can be predicted as one variable odd function of the equivalent magnetizing current for 'p' harmonic $i_{\mu,p}$:

$$E_{og}(i_{\mu,p}, p\eta) = E_0(i_{\mu,p}) \quad (3.11)$$

For mono-harmonic salient pole machines with poles on a stator or a rotor side, the co-energy function should be predicted as two variables function: an odd function of the equivalent magnetizing current for 'p' harmonic $i_{\mu,p}$ and a periodic function of an angle $p\eta$, calculated with respect to the axis of symmetry of a stator or a rotor side, respectively:

$$E_{og}(i_{\mu,p}, p\eta) = E_0(i_{\mu,p}) + E_2(i_{\mu,p})\cdot\cos 2p\eta + \cdots \quad (3.12)$$

For machines having salient poles on both sides and mono-harmonic windings, the co-energy function depends on three variables: as before on the equivalent magnetizing current for 'p' harmonic $i_{\mu,p}$ and on two angles $p\eta$ and $\varphi - p\eta$, when the angle $p\eta$ is measured with respect to the symmetry axis of a stator side or on angles $p\eta$ and $\varphi + p\eta$, when the angle $p\eta$ is measured with respect to symmetry axis of a rotor side. It can be predicted in the forms:

$$E_{\mathrm{og}}(i_{\mu,\mathrm{p}}, \mathrm{p}\eta) = E_0(i_{\mu,\mathrm{p}}) + \sum_r \sum_s E_{2r,2s}(i_{\mu,\mathrm{p}}) \cdot \cdot \cos 2r\mathrm{p}\eta \cdot \cos 2s(\varphi - \mathrm{p}\eta) \quad (3.13a)$$

or:

$$E_{\mathrm{og}}(i_{\mu,\mathrm{p}}, p\eta) = E_0(i_{\mu,\mathrm{p}}) + \sum_r \sum_s E_{2r,2s}(i_{\mu,\mathrm{p}}) \cdot \cdot \cos 2r\mathrm{p}\eta \cdot \cos 2s(\varphi + \mathrm{p}\eta) \quad (3.13b)$$

The co-energy function can be qualitatively predicted also for some cases when higher space harmonic should be taken into account. For instance, the co-energy function for a machine with a smooth air gap considering MMFs of orders 'p' and '3p' can be predicted in the form of Fourier series:

$$E_{\mathrm{og}} = \sum_{k=0}^{\infty} E_{k3\mathrm{p}}(i_{\mu,\mathrm{p}}, i_{\mu,3\mathrm{p}}) \cdot \cos k3\mathrm{p}(\eta_{\mathrm{p}} - \eta_{3\mathrm{p}}) \quad (3.14)$$

Accounting for space harmonic 'p' and '5p' the co-energy function is of the form:

$$E_{\mathrm{og}} = \sum_{k=0}^{\infty} E_{k5\mathrm{p}}(i_{\mu,\mathrm{p}}, i_{\mu,5\mathrm{p}}) \cdot \cos k5\mathrm{p}(\eta_{\mathrm{p}} - \eta_{5\mathrm{p}}) \quad (3.15)$$

However, a heuristic qualitative prediction of co-energy function properties becomes very difficult or impossible at all when more than two space harmonic should be considered or geometry of a magnetic circuit loses the symmetry.

A quantitative determination of the co-energy functions, heuristically predicted, can be done in two steps. First, the co-energy values should be found from a numerical computation of magnetic field equations in a machine magnetic circuit for properly chosen set of winding currents and rotor positions. In the second step, the equivalent magnetizing currents and respective angles should be calculated for the same set of winding currents and rotor positions.

The approach basing on properties of the co-energy function is not commonly used yet, but it is very profitable because it allows to introduce magnetic non-linearity into the "classical" circuital model of electric machines. It is relatively simple and has a physical background by using an equivalent magnetizing currents. It should be noticed that this methodology could be used to create linear models of electric machines. Assuming magnetic linearity, the co-energy function can be predicted as a quadratic form of equivalent magnetizing currents and obtained models do not use inductances at all.

Consideration of magnetic non-linearity in circuital models of electric machines has its own history and a lot of papers have been devoted to this problem. Most often the parameters of linear models are subjected to respective currents, which in many cases leads to sufficient accuracy of models solutions with measurements carried out on real machines. However, that approach is not general enough and parameters have to be chosen for an individual problem. An approach introducing into d-q models so called "cross saturation effect" is also rather popular. It gives satisfactory results from the technical point of view, but very often the conditions (3.2) which are fundamental for proper modelling of energy conversion, are not fulfilled.

4. SOLVING OF ELECTRIC MACHINE MODELS

Generally, circuital models of electric machines are described by ordinary non-linear differential equations. This non-linearity has two sources. The first one is a magnetic non-linearity of the machine magnetic circuit, which up to now has been made from ferromagnetic materials. The other one follows from the phenomenon of electromechanical energy conversion. To convert electrical energy into mechanical, the energy stored in the magnetic field of an electric machine has to depend on some mechanical variable. But this energy is a quadratic function of a current which creates the structural non-linearity of electric machine equations. For more than one hundred years history of electric machine development, solution of full models, including a motion equation has been a serious problem. Engineers simplified the circuital models in order to apply methods of classical circuit theory. First of all, electrical and mechanical equations were separated, solving electrical equations under the assumption that mechanical motion is known, and for solving mechanical equation, it was assumed that all currents are known. It means that electrical and mechanical phenomena were treated separately. Each part of equations was linearized, mainly assuming linearity of magnetic materials, and even solved analytically. In last decades the problem of solving has practically disappeared thanks to the development of computers and specialized software. Nowadays, there are a lot of commercial software packages which can solve very large sets of non-linear differential equations at a time acceptable by engineers. This statement could be used to finish further considerations on top extreme possibilities of solving methods for electric machine models.

However, circuital models give a chance for quantitative analysis of solutions, i.e. they allow to predict same properties of electrical machines without calculations in numbers. Considering the equations in a general form shown in Appendix 3, it should be stated that mathematics does not give any possibilities to find their solutions, and, therefore, the only problem to be raised is the problem of qualitative analysis of solutions of the electrical equations in the form:

$$\frac{\mathrm{d}}{\mathrm{dt}}\left(\mathbf{L}_{\mathrm{n}}(\varphi, \mathbf{i})\,\mathbf{i}\right) + \mathbf{R} \quad \mathbf{i} = \mathbf{u} \quad (4.1)$$

For periodic voltages:

$$\mathbf{u}(t) = \mathbf{u}(t + \mathrm{T_e}) = \sum_{l=-\infty}^{\infty} \mathbf{u}_k \cdot \mathrm{e}^{\mathrm{j}k\Omega_{\mathrm{e}}t}, \text{ where: } \mathrm{T_e} = \frac{2\pi}{\Omega_{\mathrm{e}}} \quad (4.2)$$

assuming also that a rotor angular velocity is periodic also with a'priori known period:

$$\omega(t) = \omega(t + \mathrm{T_m}) = \sum_{l=-\infty}^{\infty} \omega_l \cdot \mathrm{e}^{\mathrm{j}l\Omega_\mathrm{m}t} \text{ , where: } \mathrm{T_m} = \frac{2\pi}{\Omega_\mathrm{m}} \quad (4.3)$$

the steady state solution of (4.1) can be foreseen in the form of double Fourier series:

$$\mathbf{i} = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \mathbf{I}_{k,l} \cdot \mathrm{e}^{\mathrm{j}(k\Omega_\mathrm{e} + l\Omega_\mathrm{m})t} \quad (4.4)$$

The Fourier's coefficients have to fulfill an infinite algebraic equation set, balancing the series on both sides of the equation (4.1):

$$\mathbf{U} = (\mathbf{R} + \mathrm{j}\,\Omega\mathbf{L}_\mathrm{n}(\mathbf{I})) * \mathbf{I} \quad (4.5)$$

These equations are non-linear because the elements of the $\mathbf{L}_\mathrm{n}(\varphi, \mathbf{i})$ matrix depend on a solution which is being looked for. Analyzing carefully, the structure of this set the current can be determined qualitatively. Basing on this equation, the current can be also determined quantitatively. Therefore, equations (4.1) should be solved iteratively. The simplest procedure consists in a direct iterative algorithm solving in successive iterations a set of linear equations, but limited to finite dimensions:

$$\mathbf{U} = (\mathbf{R} + \mathrm{j}\,\Omega\mathbf{L}_\mathrm{n}(\mathbf{I}_i)) * \mathbf{I}_{i+1} \quad (4.6)$$

For arrangement of Fourier coefficients $\mathbf{I}_{k,l}$ of the current vector, according to the notation given below:

$$\mathbf{I} = \begin{bmatrix} \cdots & \mathbf{I}_{-1} & \mathbf{I}_0 & \mathbf{I}_1 & \cdots \end{bmatrix}^T$$
$$\mathbf{I}_k = \begin{bmatrix} \cdots & \mathbf{I}_{k,-1} & \mathbf{I}_{k,0} & \mathbf{I}_{k,1} & \cdots \end{bmatrix}^T \quad (4.7)$$

the matrices in the equation (4.6) take the forms:

$$\mathbf{R} = \mathrm{diag}\begin{bmatrix} \cdots & \mathbf{R} & \mathbf{R} & \mathbf{R} & \cdots \end{bmatrix}$$
$$\mathbf{R} = \mathrm{diag}\begin{bmatrix} \cdots & \mathbf{R} & \mathbf{R} & \mathbf{R} & \cdots \end{bmatrix}$$
$$\mathbf{\Omega} = \mathrm{diag}\begin{bmatrix} \cdots & \Omega_{-1} & \Omega_0 & \Omega_1 & \cdots \end{bmatrix} \quad (4.8)$$

$$\mathbf{\Omega}_k = \mathrm{diag}\begin{bmatrix} \cdots & k\,\Omega_\mathrm{e} - \Omega_\mathrm{m} & k\,\Omega_\mathrm{e} & k\,\Omega_\mathrm{e} + \Omega_\mathrm{m} & \cdots \end{bmatrix} \quad (4.9)$$

$$\mathbf{L}_\mathrm{n}(\mathbf{I}) = \begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & \mathbf{L}_0 & \mathbf{L}_1 & \mathbf{L}_2 & \cdots \\ \cdots & \mathbf{L}_{-1} & \mathbf{L}_0 & \mathbf{L}_1 & \cdots \\ \cdots & \mathbf{L}_{-2} & \mathbf{L}_{-1} & \mathbf{L}_0 & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix} \quad (4.10)$$

$$\mathbf{L}_k = \begin{bmatrix} & \vdots & \vdots & \vdots & \\ \cdots & \mathbf{L}_{k,0} & \mathbf{L}_{k,1} & \mathbf{L}_{k,2} & \cdots \\ \cdots & \mathbf{L}_{k,-1} & \mathbf{L}_{k,0} & \mathbf{L}_{k,1} & \cdots \\ \cdots & \mathbf{L}_{k,-2} & \mathbf{L}_{k,-1} & \mathbf{L}_{k,0} & \cdots \\ & \vdots & \vdots & \vdots & \end{bmatrix} \quad (4.10)$$

where $\mathbf{L}_{k,l}$ are the Fourier coefficients of the matrix $\mathbf{L}_\mathrm{n}(\varphi, \mathbf{i})$:

$$\mathbf{L}_\mathrm{n}(\varphi, \mathbf{i}) = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \mathbf{L}_{k,l} \cdot \mathrm{e}^{\mathrm{j}(k\Omega_\mathrm{e} + l\Omega_\mathrm{m})t} \quad (4.11)$$

substituting in it the time function of the angle and time functions of currents obtained in a previous iteration. A vector of voltages $\mathbf{U}$ in the detailed notation takes the from:

$$\mathbf{U} = \begin{bmatrix} \cdots & \mathbf{U}_{-1} & \mathbf{U}_0 & \mathbf{U}_1 & \cdots \end{bmatrix}^T$$
$$\mathbf{U}_k = \begin{bmatrix} \cdots & \mathbf{0} & \mathbf{u}_k & \mathbf{0} & \cdots \end{bmatrix}^T \quad (4.12)$$

For solving non-linear equation set (4.5) Newton-Raphson algorithm can be also used. It requires to solve in the successive iteration the equation set of the form

$$\mathbf{U} + \mathrm{j}\Omega(\mathbf{L}_\mathrm{d}(\mathbf{I}_i) - \mathbf{L}_\mathrm{n}(\mathbf{I}_i)) * \mathbf{I}_i = (\mathbf{R} + \mathrm{j}\,\Omega\mathbf{L}_\mathrm{d}(\mathbf{I}_i)) * \mathbf{I}_{i+1} \quad (4.13)$$

In this equation there appears the matrix $\mathbf{L}_\mathrm{d}(\mathbf{I})$, which is created analogously to the matrix $\mathbf{L}_\mathrm{n}(\mathbf{I})$, but on the basis of the Fourier series of a dynamic inductance matrix:

$$\mathbf{L}_\mathrm{d}(\varphi, \mathbf{i}) = \frac{\partial\, \Psi(\varphi, \mathbf{i})}{\partial\, \mathbf{i}} = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \mathbf{L}^\mathrm{d}_{k,l} \cdot \mathrm{e}^{\mathrm{j}(k\Omega_\mathrm{e} + l\Omega_\mathrm{m})t} \quad (4.14)$$

where $\Psi(\varphi, \mathbf{i}) = \mathbf{L}_\mathrm{n}(\varphi, \mathbf{i}) \cdot \mathbf{i}$. The Newton-Raphson algorithm is much faster converged.

To solve both the equations (4.6) as well as the equation (4.13), the Fourier coefficients of the Fourier series (4.11) of matrix $\mathbf{L}_\mathrm{n}(\varphi, \mathbf{i})$ and (4.14) of the matrix $\mathbf{L}_\mathrm{d}(\varphi, \mathbf{i})$ have to be calculated in each iteration. Elements of these matrices are non-linear functions of currents and the rotation angle, so calculation of those Fourier coefficients have to be done in three steps. First, the values of all currents and the angle are calculated from the Fourier series obtained in previous iteration for a set of chosen time instant. Next, the time functions of all elements of the matrices $\mathbf{L}_\mathrm{n}(\varphi, \mathbf{i})$ and $\mathbf{L}_\mathrm{d}(\varphi, \mathbf{i})$, are calculated in the same set of time instant, and finally, new values of the Fourier coefficients of the series (4.11) and (4.14), required in the equations (4.5) or (4.13) can be found.

The electromagnetic torque of an electric machine can be obtained from the general formula presented in Appendix 3. It is described by a quadratic form:

$$T_\mathrm{em} = \mathbf{i}^T \cdot \mathbf{W}(\varphi, \mathbf{i}) \cdot \mathbf{i} \quad (4.15)$$

The matrix of this form depends on the currents and the rotation angle too, so it can be written in the form of double series, like the matrices $\mathbf{L}_\mathrm{n}(\varphi, \mathbf{i})$ and $\mathbf{L}_\mathrm{d}(\varphi, \mathbf{i})$:

$$\mathbf{W}(\varphi,\mathbf{i}) = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \mathbf{W}_{k,l} \cdot \mathrm{e}^{\mathrm{j}(k\Omega_\mathrm{e}+l\Omega_\mathrm{m})t} \qquad (4.16)$$

The formula for electromagnetic torque can be presented in the double Fourier series:

$$T_\mathrm{em} = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} \mathbf{T}_{k,l} \cdot \mathrm{e}^{\mathrm{j}(k\Omega_\mathrm{e}+l\Omega_\mathrm{m})t} \qquad (4.17)$$

and its coefficients can be written as the quadratic forms of the Fourier coefficients of currents [9, 11].

The method of steady state analysis presented above, has really extreme possibilities, but in a general case is rather complicated. For simpler cases, i.e. for models of machines having same internal symmetry, this method is very effective and allows to predict qualitative properties of both currents and the electromagnetic torque, and it also allows calculating their values. Such an analysis is more profound than a determination of machine properties from numerical solutions of differential equations. However, such a tendency is nowadays observed.

5. CONCLUSIONS

The ideas presented in the paper reflect the author's opinions on extreme possibilities of circuital models of electric machines. For more than one hundred year history of development of electric machines these models have been dominant and the knowledge about them is enormous. Therefore, the opinions presented in the paper have to be selective. They have become limited to the critical problems of mathematical modeling of electric machines: generating of equations of linear models with the highest degree of complexity, simplification of these equations, generating of equations which take into account magnetic non-linearity, and the most general method of predicting steady state solutions.

The increasing role of field models makes circuital models be confronted with new tasks, which involves their close relations to numerical analysis of electromagnetic field in a machine. Circuital models, however, should not be considered to be in competition with field models, but rather as their essential supplement. There is no doubt that circuital models of electric machines will continue for a long time to be the basic tool to investigate phenomena in machines, especially, to research their performance with more and more complex co-operating systems.

REFERENCES

1. Adkins B., Harley R.G.: *The General Theory of Alternating Current Machines*. London, Chapman Hall, 1975.
2. Concordia C.H.: *Synchronous Machines, Theory and Performance*. J. Wiley & Sons Inc., New York, 1951.
3. Kovacs K.P., Racz I.: *Transiente Vorgange in Wechselstrommaschinen*. Verlag der Ungarische Akademie der Wissenschaften, 1959.
4. Kluszczynski K., Miksiewicz R.: *Parasitic torque of cage induction motors* (in Polish). Pub. by PTETiS, Warsaw-Gliwice, 1993.
5. Paszek W.: *Transients of Alternating Current Machines* (in Polish). WNT, Warsaw, 1986.
6. Paszek W.: *Alternating Current Machine Dynamics* (in Polish). Helion, Gliwice, 1998.
7. Puchała A.: *Dynamics of Electric Machines and Electromechanical Systems* (in Polish). Warsaw, WNT, 1977.
8. Vas P.: *Electric Machines and Drives: A Space-Vector Theory Approach*. Oxford University Press, 1992.
9. Rusek J.: *Computer Analysis of Induction Machines by Harmonics Balance*. Pub. by Univ. of Mining & Metallurgy, Cracow, 2000.
10. Skwarczyński J.: *Electric Machines* (in Polish), Parts. I i II. Pub. by Univ. of Mining & Metallurgy, Cracow,1997.
11. Sobczyk T.J.: *Methodology of Mathematical Modeling of Induction Machines* (in Polish). WNT, Warsaw, 2004.
12. White D.C., Woodson H.H.: *Electromechanical Energy Conversion*. J. Wiley & Sons Inc., New York, 1959.

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Appendix 1 - Inductance matrices of a base model

$$\mathbf{L}^{m}{}_{ss}(\varphi)=\sum_{s}\sum_{r}\left(\operatorname{diag}\begin{bmatrix} e^{jr\alpha_1} \\ e^{jr\alpha_2} \\ \vdots \\ e^{jr\alpha_M} \end{bmatrix}\sum_{\nu}\begin{bmatrix} L^{1,1}_{\nu,r,s} & L^{1,2}_{\nu,r,s}e^{j\nu(\alpha_1-\alpha_2)} & \cdots & L^{1,M}_{\nu,r,s}e^{j\nu(\alpha_1-\alpha_M)} \\ L^{2,1}_{\nu,r,s}e^{j\nu(\alpha_2-\alpha_1)} & L^{2,2}_{\nu,r,s} & \cdots & L^{2,M}_{\nu,r,s}e^{j\nu(\alpha_2-\alpha_M)} \\ \vdots & \vdots & \ddots & \vdots \\ L^{M,1}_{\nu,r,s}e^{j\nu(\alpha_M-\alpha_!)} & L^{M,2}_{\nu,r,s}e^{j\nu(\alpha_M-\alpha_2)} & \cdots & L^{M,M}_{\nu,r,s} \end{bmatrix}e^{js\varphi}\right)$$

$$\mathbf{L}^{m}{}_{rr}(\varphi)=\sum_{s}\sum_{r}\left(\operatorname{diag}\begin{bmatrix} e^{jr\beta_1} \\ e^{jr\beta_2} \\ \vdots \\ e^{jr\beta_N} \end{bmatrix}\sum_{\nu}\begin{bmatrix} L^{M+1,M+1}_{\nu,r,s} & L^{M+1,M+2}_{\nu,r,s}e^{j\nu(\beta_1-\beta_2)} & \cdots & L^{M+1,M+N}_{\nu,r,s}e^{j\nu(\beta_1-\beta_N)} \\ L^{M+2,M+1}_{\nu,r,s}e^{j\nu(\beta_2-\beta_1)} & L^{M+2,M+2}_{\nu,r,s} & \cdots & L^{M+2,M+N}_{\nu,r,s}e^{j\nu(\beta_2-\beta_N)} \\ \vdots & \vdots & \ddots & \vdots \\ L^{M+N,M+1}_{\nu,r,s}e^{j\nu(\beta_N-\beta_1)} & L^{M+N,M+2}_{\nu,r,s}e^{j\nu(\beta_N-\beta_2)} & \cdots & L^{M+N,M+N}_{\nu,r,s} \end{bmatrix}e^{j(r+s)\varphi}\right)$$

$$\mathbf{L}^{m}{}_{rs}(\varphi)=\sum_{s}\sum_{r}\left(\operatorname{diag}\begin{bmatrix} e^{jr\beta_1} \\ e^{jr\beta_2} \\ \vdots \\ e^{jr\beta_N} \end{bmatrix}\sum_{\nu}\begin{bmatrix} L^{M+1,1}_{\nu,r,s}e^{j\nu(\beta_1-\alpha_1)} & L^{M+1,2}_{\nu,r,s}e^{j\nu(\beta_1-\alpha_2)} & \cdots & L^{M+1,N}_{\nu,r,s}e^{j\nu(\beta_1-\alpha_M)} \\ L^{M+2,1}_{\nu,r,s}e^{j\nu(\beta_2-\alpha_1)} & L^{M+2,2}_{\nu,r,s}e^{j\nu(\beta_2-\alpha_2)} & \cdots & L^{M+2,N}_{\nu,r,s}e^{j\nu(\beta_2-\alpha_M)} \\ \vdots & \vdots & \ddots & \vdots \\ L^{M+N,1}_{\nu,r,s}e^{j\nu(\beta_N-\alpha_1)} & L^{M+N,2}_{\nu,r,s}e^{j\nu(\beta_N-\alpha_2)} & \cdots & L^{M+N,N}_{\nu,r,s}e^{j\nu(\beta_N-\alpha_M)} \end{bmatrix}e^{j(\nu+r+s)\varphi}\right)$$

Appendix 2 - Inductance matrices of a base model for symmetrically located coils

$$\mathbf{L}^{m}{}_{ss}=\begin{bmatrix} L_s(0,0,\varphi) & L_s(-\alpha,0,\varphi) & \cdots & L_s(\alpha,0,\varphi) \\ L_s(\alpha,\alpha,\varphi) & L_s(0,\alpha,\varphi) & \cdots & L_s(2\alpha,\alpha,\varphi) \\ \vdots & \vdots & \ddots & \vdots \\ L_s(-\alpha,-\alpha,\varphi) & L_s(-2\alpha,-\alpha,\varphi) & \cdots & L_s(0,-\alpha,\varphi) \end{bmatrix}$$

$$\mathbf{L}^{m}{}_{rr}=\begin{bmatrix} L_r(0,\varphi,\varphi) & L_r(-\beta,\varphi,\varphi) & \cdots & L_r(\beta,\varphi,\varphi) \\ L_r(\beta,\varphi+\beta,\varphi) & L_r(0,\varphi+\beta,\varphi) & \cdots & L_r(2\beta,\varphi+\beta,\varphi) \\ \vdots & \vdots & \ddots & \vdots \\ L_r(-\beta,\varphi-\beta,\varphi) & L_r(-2\beta,\varphi-\beta,\varphi) & \cdots & L_r(0,\varphi-\beta,\varphi) \end{bmatrix}$$

$$\mathbf{L}^{m}{}_{rs}=\begin{bmatrix} L_{rs}(\varphi,\varphi,\varphi) & L_{rs}(\varphi-\alpha,\varphi,\varphi) & \cdots & L_{rs}(\varphi+\alpha,\varphi,\varphi) \\ L_{rs}(\varphi+\beta,\varphi+\beta,\varphi) & L_{rs}(\varphi+\beta-\alpha,\varphi+\beta,\varphi) & \cdots & L_{rs}(\varphi+\beta+\alpha,\varphi+\beta,\varphi) \\ \vdots & \vdots & \ddots & \vdots \\ L_{rs}(\varphi-\beta,\varphi-\beta,\varphi) & L_{rs}(\varphi-\beta-\alpha,\varphi-\beta,\varphi) & \cdots & L_{rs}(\varphi-\beta+\alpha,\varphi-\beta,\varphi) \end{bmatrix}$$

Appendix 3 - General equations of circiutal models accounting for non-linearity of a main magnetic circuit

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\psi_{\sigma,1}\\ \psi_{\sigma,2}\\ \vdots\\ \psi_{\sigma,\mathrm{N}}\end{bmatrix}+\frac{\mathrm{d}}{\mathrm{d}t}\left\{\sum_{\rho}\frac{1}{i_{\mu\rho}}\frac{\partial E_{og}}{\partial i_{\mu\rho}}\cdot\begin{bmatrix}(\nu_1)^2 & (\nu_1\nu_2)\cos\rho(\alpha_1-\alpha_2) & \cdots & (\nu_1\nu_\mathrm{N})\cos\rho(\alpha_1-\alpha_\mathrm{N})\\ (\nu_2\nu_1)\cos\rho(\alpha_2-\alpha_1) & (\nu_2)^2 & \cdots & (\nu_2\nu_\mathrm{N})\cos\rho(\alpha_2-\alpha_\mathrm{N})\\ \vdots & \vdots & \ddots & \vdots\\ (\nu_\mathrm{N}\nu_1)\cos\rho(\alpha_\mathrm{N}-\alpha_1) & (\nu_\mathrm{N}\nu_2)\cos\rho(\alpha_\mathrm{N}-\alpha_2) & \cdots & (\nu_\mathrm{N})^2\end{bmatrix}\begin{bmatrix}i_1\\ i_2\\ \vdots\\ i_\mathrm{N}\end{bmatrix}\right\}+$$

$$+\frac{\mathrm{d}}{\mathrm{d}t}\left\{\sum_{\rho}\frac{1}{(i_{\mu\rho})^2}\frac{\partial E_{og}}{\partial(\rho\eta)}\cdot\begin{bmatrix}0 & (\nu_1\nu_2)\sin\rho(\alpha_1-\alpha_2) & \cdots & (\nu_1\nu_\mathrm{N})\sin\rho(\alpha_1-\alpha_\mathrm{N})\\ (\nu_2\nu_1)\sin\rho(\alpha_2-\alpha_1) & 0 & \cdots & (\nu_2\nu_\mathrm{N})\sin\rho(\alpha_2-\alpha_\mathrm{N})\\ \vdots & \vdots & \ddots & \vdots\\ (\nu_\mathrm{N}\nu_1)\sin\rho(\alpha_\mathrm{N}-\alpha_1) & (\nu_\mathrm{N}\nu_2)\sin\rho(\alpha_\mathrm{N}-\alpha_2) & \cdots & 0\end{bmatrix}\begin{bmatrix}i_1\\ i_2\\ \vdots\\ i_\mathrm{N}\end{bmatrix}\right\}=$$

$$=\begin{bmatrix}u_1\\ u_2\\ \vdots\\ u_\mathrm{N}\end{bmatrix}-\begin{bmatrix}\mathrm{R}_1 & & & \\ & \mathrm{R}_2 & & \\ & & \ddots & \\ & & & \mathrm{R}_\mathrm{N}\end{bmatrix}\begin{bmatrix}i_1\\ i_2\\ \vdots\\ i_\mathrm{N}\end{bmatrix}$$

$$\mathrm{J}\frac{\mathrm{d}^2\varphi}{\mathrm{d}t^2}=\tfrac{1}{2}\sum_{\rho}\frac{1}{i_{\mu\rho}}\frac{\partial E_{og}}{\partial i_{\mu\rho}}\cdot\begin{bmatrix}i_1 & i_2 & \cdots & i_\mathrm{N}\end{bmatrix}\left\{\frac{\partial}{\partial\varphi}\begin{bmatrix}(\nu_1)^2 & (\nu_1\nu_2)\cos\rho(\alpha_1-\alpha_2) & \cdots & (\nu_1\nu_\mathrm{N})\cos\rho(\alpha_1-\alpha_\mathrm{N})\\ (\nu_2\nu_1)\cos\rho(\alpha_2-\alpha_1) & (\nu_2)^2 & \cdots & (\nu_2\nu_\mathrm{N})\cos\rho(\alpha_2-\alpha_\mathrm{N})\\ \vdots & \vdots & \ddots & \vdots\\ (\nu_\mathrm{N}\nu_1)\cos\rho(\alpha_\mathrm{N}-\alpha_1) & (\nu_\mathrm{N}\nu_2)\cos\rho(\alpha_\mathrm{N}-\alpha_2) & \cdots & (\nu_\mathrm{N})^2\end{bmatrix}\right\}\begin{bmatrix}i_1\\ i_2\\ \vdots\\ i_\mathrm{N}\end{bmatrix}+$$

$$+\sum_{\rho}\frac{1}{(i_{\mu\rho})^2}\frac{\partial E_{og}}{\partial(\rho\eta)}\cdot\begin{bmatrix}i_1 & i_2 & \cdots & i_\mathrm{N}\end{bmatrix}\begin{bmatrix}\frac{\partial\alpha_1}{\partial\varphi} & & & \\ & \frac{\partial\alpha_2}{\partial\varphi} & & \\ & & \ddots & \\ & & & \frac{\partial\alpha_\mathrm{N}}{\partial\varphi}\end{bmatrix}\cdot$$

$$\cdot\begin{bmatrix}(\nu_1)^2 & (\nu_1\nu_2)\cos\rho(\alpha_1-\alpha_2) & \cdots & (\nu_1\nu_\mathrm{N})\cos\rho(\alpha_1-\alpha_\mathrm{N})\\ (\nu_2\nu_1)\cos\rho(\alpha_2-\alpha_1) & (\nu_2)^2 & \cdots & (\nu_2\nu_\mathrm{N})\cos\rho(\alpha_2-\alpha_\mathrm{N})\\ \vdots & \vdots & \ddots & \vdots\\ (\nu_\mathrm{N}\nu_1)\cos\rho(\alpha_\mathrm{N}-\alpha_1) & (\nu_\mathrm{N}\nu_2)\cos\rho(\alpha_\mathrm{N}-\alpha_2) & \cdots & (\nu_\mathrm{N})^2\end{bmatrix}\begin{bmatrix}i_1\\ i_2\\ \vdots\\ i_\mathrm{N}\end{bmatrix}+T_\mathrm{m}$$